Communication Theory
Summary of Important Definitions and Results

Signal and system theory

Convolution of signals \( x(t) * h(t) = y(t) \):

\[
y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t - \lambda) \, d\lambda.
\]

Fourier Transform:

\[
X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt, \quad \text{for all signals}
\]

\[
X(\omega) = 2 \int_{0}^{\infty} x(t) \cos(\omega t) \, dt, \quad \text{for even signals}
\]

\[
X(\omega) = -2j \int_{0}^{\infty} x(t) \sin(\omega t) \, dt, \quad \text{for odd signals}
\]

Properties of the Fourier Transform:

If \( x(t) \leftrightarrow X(\omega) \) and \( v(t) \leftrightarrow V(\omega) \), then:

\[
ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)
\]

\[
x(t - t_0) \leftrightarrow X(\omega)e^{-j\omega t_0}
\]

\[
x(-t) \leftrightarrow X(-\omega)
\]

\[
x(at) \leftrightarrow \frac{1}{a}X\left(\frac{\omega}{a}\right), \quad \text{if } a > 0
\]

\[
t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n}X(\omega)
\]

\[
x(t) e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0), \quad \text{if } \omega_0 \text{ is real}
\]

\[
x(t) \cos(\omega_0 t) \leftrightarrow \frac{1}{2} [X(\omega + \omega_0) + X(\omega - \omega_0)]
\]

\[
x(t) \sin(\omega_0 t) \leftrightarrow \frac{j}{2} [X(\omega + \omega_0) - X(\omega - \omega_0)]
\]

\[
\frac{d^n}{dt^n}x(t) \leftrightarrow (j\omega)^nX(\omega)
\]

\[
\int_{-\infty}^{t} x(\lambda) \, d\lambda \leftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)
\]

\[
x(t) * v(t) \leftrightarrow X(\omega)V(\omega)
\]

\[
x(t)v(t) \leftrightarrow \frac{1}{2\pi}X(\omega) * V(\omega)
\]

\[
X(t) \leftrightarrow 2\pi x(-\omega)
\]
Common Fourier Transform Pairs:

\[ \delta(t) \leftrightarrow 1 \]
\[ u(t) \leftrightarrow \pi \delta(\omega) + \frac{1}{j\omega} \]
\[ \cos(\omega_0 t) \leftrightarrow \pi \left[ \delta(\omega + \omega_0) + \delta(\omega - \omega_0) \right] \]
\[ \sin(\omega_0 t) \leftrightarrow j\pi \left[ \delta(\omega + \omega_0) - \delta(\omega - \omega_0) \right] \]

Autocorrelation function for deterministic signals:

\[ R_x(\tau) = \int_{-\infty}^{\infty} x(t)x(t + \tau) \, dt, \text{ for an energy signal } x(t). \]
\[ R_x(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)x(t + \tau) \, dt, \text{ for a power signal } x(t) \text{ with period } T_0. \]

Sampling and Quantization

\[ x_\delta(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s), \text{ is a periodic impulse train of period } T_s. \]
\[ x_\delta(t) \leftrightarrow \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T_s}). \]

SNR_q = 3L^2, is the quantization SNR with an \( L \)-level quantizer.

Digital modulation and demodulation

\[ Q(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{2}\right) \, dt. \]

If \( a_1 \) and \( a_2 \) are the two possible sample values at the output of a digital receiver and \( \sigma_0^2 \) is the variance of the output noise, the bit error probability \( P_B \) is:

\[ P_B = Q\left(\frac{a_1 - a_2}{2\sigma_0}\right). \]

When a matched filter is used, the error probability depends on the energy \( E_d \) of the transmission signal-difference and the power spectral density \( N_0/2 \) of the Gaussian noise as:

\[ P_B = Q\left(\frac{\sqrt{E_d}}{2N_0}\right). \]

The raised cosine (RC) filter with a bandwidth \( W \) and transmission rate \( 1/T = 2W_0 \) has the following characteristic:

\[ H(f) = \begin{cases} 
1 & \text{for } |f| < 2W_0 - W \\
\cos^2 \left( \frac{\pi}{4} \frac{|f| + W - 2W_0}{W - W_0} \right) & \text{for } 2W_0 - W < |f| < W \\
0 & \text{for } |f| > W 
\end{cases} \]
If the roll-off factor of the raised cosine pulse is $r$, then,

$$ W = \frac{(1 + r)}{2T} $$

**M-ary modulation and demodulation**

Symbol error probability of $M$-ary phase shift keying (MPSK) with energy $E_s$ per symbol in the presence of Gaussian noise with power spectral density $N_0/2$ Watts/Hz is

$$ P_E(M) \approx 2Q \left( \frac{2E_s}{N_0} \sin\left(\frac{\pi}{M}\right) \right). $$

Bit error probability is $P_B = P_E / \log_2 M$.

Symbol error probability of non-coherently detected $M$-ary frequency shift keying (MFSK) with energy $E_s$ per symbol in the presence of Gaussian noise with power spectral density $N_0/2$ Watts/Hz is

$$ P_E(M) \approx \frac{M - 1}{2} \exp\left( -\frac{E_s}{2N_0} \right). $$

Bit error probability is $P_B = \frac{M/2}{M-1} P_E$.

**Communication link analysis**

The power $P_r$ received by the receiver with antenna gain $G_r$ at a distance $d$ away from a transmitter transmitting an electromagnetic signal (of wavelength $\lambda$) a power $P_t$ with an antenna gain of $G_t$ is:

$$ P_r = \frac{P_t G_t G_r \lambda^2}{(4\pi d)^2} $$

The gain $G$ of an antenna is related to its effective area $A_e$ and wavelength $\lambda$ of transmission as:

$$ G = \frac{4\pi A_e}{\lambda^2}. $$

The link margin $M$ of a communication link is related to the effective isotropic radiated power (EIRP), receiver antenna gain $G_r$, required signal to noise ratio $(E_b/N_0)\text{reqd}$, data rate $R$, Boltzmann’s constant $k$, free space loss $L_s$, other incidental losses $L_0$ and temperature $T^\circ$ (in Kelvin) as

$$ M = \frac{\text{EIRP} G_r}{(E_b/N_0)\text{reqd} R k T^\circ L_s L_0} $$

The noise figure $F$ and noise temperature $T^\circ$ of an amplifier are related as

$$ T^\circ = (F - 1)290. $$

The noise figure $F$ of a lossy line with loss factor $L$ is

$$ F = L. $$
The composite noise figure $F_{\text{comp}}$ of a cascade of $n$ amplifiers with gains $G_i$ and noise figures $F_i$ is:

$$F_{\text{comp}} = F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1G_2} + \ldots + \frac{F_n - 1}{G_1G_2 \ldots G_{n-1}}$$

The effective bit error probability $P_B$ of a regenerative satellite repeater link is related to the bit error probabilities $P_u$ and $P_d$ of the uplink and downlink, respectively, as

$$P_B \approx P_u + P_d.$$ 

The received signal-to-noise ratio $(Pr/N_0)_{ij}$ at the $j$th receiver from the $i$th transmitter (with power $P_i$) through a non-regenerative satellite repeater with total isotropic power EIRP$_s$ can be computed using $\gamma_j$ the downlink loss, $A_i$ the uplink loss, $N_s/2$ the Gaussian noise power spectral density at the satellite, $N_g/2$ the Gaussian noise power spectral density at the receiver, $P_T$ the total power received at the satellite, $W$ the bandwidth, and, $\beta = 1/(P_T + N_sW)$ as:

$$\left(\frac{Pr}{N}\right)_{ij} = \frac{\text{EIRP}_s \gamma_j \beta A_i P_i}{\text{EIRP}_s \gamma_j \beta N_s + N_g}$$

**Multiple access techniques**

The normalized throughput $\rho$ is related to the normalized traffic $G$ on a pure ALOHA channel as:

$$\rho = Ge^{-2G}$$

and has a maximum at $G = 0.5$ when $\rho = 0.18$.

The normalized throughput $\rho$ is related to the normalized traffic $G$ on a slotted ALOHA channel as:

$$\rho = Ge^{-G}$$

and has a maximum at $G = 1$ when $\rho = 0.37$.

**Spread spectrum techniques**

The bit error probability $P_B$ of non-coherent BFSK with a partial band jammer that jams a fraction $\rho$ of the spread bandwidth $W_{ss}$ is related to the energy per bit $E_b$ and jammer power spectral density $J_0$ over the bandwidth as:

$$P_B \approx \frac{\rho}{2} \exp \left( -\frac{\rho E_b}{2J_0} \right),$$

and the worst case jamming occurs when $\rho = \frac{2}{E_b/J_0}$, assuming that $E_b/J_0 > 2$.

The bit error probability $P_B$ of BPSK with a pulse jammer that jams at a duty cycle of $\rho$ is related to the energy per bit $E_b$ and jammer power spectral density $J_0$ over the spread bandwidth $W_{ss}$ as:

$$P_B \approx \rho Q \left( \sqrt{\frac{2\rho E_b}{J_0}} \right),$$
and the worst case jamming occurs when $\rho = \frac{0.709}{E_b/J_0}$, assuming that $E_b/J_0 > 0.709$.

The maximum number $M$ of users that can be accommodated by a spread spectrum system is related to the spreading gain $G_p$, the voice activity factor $G_V$, antenna gain $G_A$, asynchronism factor $\gamma$, outer cell interference factor $H_0$, and the signal to interference ratio $(E_b/I_0)_{reqd}$ required for acceptable performance as:

$$M = \frac{\gamma G_p G_A G_V}{(E_b/I_0)_{reqd} H_0}.$$