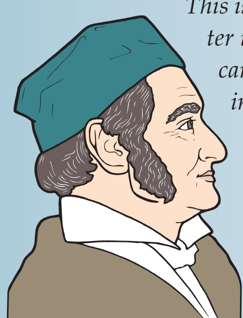
Speaker's Corner

This issue's Speaker's Corner features two items. The first is a Gaussian fable authored by Madhu S. Gupta, who will become the new editor-in-chief of this *Magazine* starting in 2003. The second, on page 25, is a report on AdCom-approved changes to the MTT-S bylaws by Michael DeLisio.

A Fable from Vector Valley

■ Madhu S. Gupta



Carl Friedrich Gauss

This is a fable about locating submerged sources and sinks of water in a flooded valley, based solely on flow rate measurements carried out around closed boundaries. Aimed at students in introductory courses on vector calculus, electromagnetism, and continuum mechanics, it is meant to motivate vector integration, couch it in physical terms, make it intuitively reasonable, exemplify its use, develop some understanding of surface integrals, and at the same time hold the students' interest by being entertaining. Quantitative basis for the fable and links to a discussion of Gauss' law in electrostatics are presented.

Once upon a time, in a valley at the foot of a mountain, lived a happy community with their homes, ranches, and farms neatly arranged in arrays of fenced yards. One eventful day, there was an earthquake in the valley, and it suddenly flooded with water. The water rose quickly, and soon most of the valley was under varying heights of standing water, ankle-deep in some places and neck-deep in others. Thereafter, the level of water no longer changed with time, although the water itself was not stationary, and continued to flow with speeds and in directions that varied from place to place, but not with time. You might say a time-invariant, or "static," state had been reached.

The source of this water was a mystery to everyone, and there was much speculation as to the source, ranging from the mystical (angry gods) to technical (a rupture in the underground aqueduct that passed through the valley). A likely explanation was that the earthquake had opened up some hitherto underground spring. Since the water level was static, some thought that no "new" water was being added, and the water was merely circulating. Others explained the flow of water by postulating that the earthquake must have opened up not only a spring but also some fissures in the ground that were draining away the gushing water. The perplexed officials at the city hall decided to send someone to find out where all this water was coming from or going to. They selected an engineer by the name of Gar Clauss for the investigation.

Being a brave engineer, Gar Clauss was willing to take the usual risks of moving around in standing flood waters, such as rapid undercurrents in the water, shifting ground, and snakes or other dangerous animals that might be found in flood waters. So he began surveying the individual properties in the valley to find where the source(s) and sink(s) of the water were. This, he discovered, was not so easy—many large properties in the valley were surrounded by a chain-link fence that he could not enter due to absentee owners, trespassing laws, or secretive residents, not to mention the prospects of meeting an occasional ferocious dog guarding the fence. To get a court order to enter a private fenced area required sufficient grounds to suspect that the source or sink of water lay within it. A visual examination from the outside was not helpful since any sources and sinks of water were submerged. Consequently, any investigations that Gar Clauss could make had to be based on the observed flow of floodwater, and that observation could only be carried out from outside each property around the periphery of its fence!

M.S. Gupta is with San Diego State University in San Diego, California, USA.

Quantitative Basis of the Fable

Fables are said to be as old as mankind and have long been used to convey lessons, morals, and teachings. They can be effective in other teaching situations as well, such as in clarifying complex or conceptual issues. Vector calculus involves many such conceptual difficulties for novices where a physical crutch can be helpful.

The recounted fable is based on the well known "continuity equation," sometimes also called the Law of Conservation. This equation is essentially a bookkeeping equation used to track the whereabouts of a physical quantity that is capable of flowing like a fluid; examples are fluid volume, number (or density) of particles, mass, charge, and energy. The flow of such a quantity is quantitatively described by a flux (i.e., the rate of transfer across a given surface) and represented by a point vector $\boldsymbol{J}$, because it has both a magnitude and a direction at each point. The surface integral of the flux $\boldsymbol{J}$ of that physical quantity, carried out over the closed surface S, describes the net rate of outflow of that quantity from the closed surface S, and, therefore, must be related to the rate at which that quantity is produced (created, generated, introduced, annihilated, removed, or stored away) within the closed surface S by some source, agent, or process; however, if the physical quantity is conserved, a net outflow must cause a change in the volume density ρ of the physical quantity within the volume V enclosed by the closed surface S, leading to the continuity equation

$$\int_S \boldsymbol{J} \cdot d\boldsymbol{S} = -d\left(\int_V \rho dV\right)\Big/dt. \tag{1}$$

This equation is a physical statement about the nature of the flowing quantity, since it is applicable only if the physical quantity involved is extensive (i.e., additive) and conserved (e.g., fluid volume, provided the fluid is incompressible). If the surface S is fixed, so that the flow occurs only due to the movement of the quantity rather than the surface, only that component of the flow need be accounted for, so that

$$\int_S \boldsymbol{J} \cdot d\boldsymbol{S} = -\partial\left(\int_V \rho dV\right)\Big/\partial t. \tag{2}$$

This so called "integral form" of the continuity equation is the basis of the fable recounted here.

A second form of the same equation can be arrived at through the use of divergence theorem, due to Gauss, sometimes also called the Gauss-Ostrogradsky theorem. This famous theorem relates the surface integral of a vector field $\boldsymbol{A}$, carried out over a closed surface S, to the volume integral of the divergence of that vector field over a volume V enclosed by that closed surface.

$$\int_S \boldsymbol{A} \cdot d\boldsymbol{S} = \int_V (\nabla \cdot \boldsymbol{A}) dV. \tag{3}$$

This theorem is a mathematical identity in vector calculus and is applicable to any vector field, subject only to some subtle conditions concerning smoothness of the vector field $\boldsymbol{A}$, convergence of integrals, and connectedness of the closed surface.

When the divergence theorem (3) is applied to the flux $\boldsymbol{J}$, which is a vector field, in (2) in the limiting case of a small volume, it leads to a differential (or point or local) form of the continuity equation,

$$\nabla \cdot \boldsymbol{J} = -\partial\rho/\partial t. \tag{4}$$

The two forms of continuity equations (2) and (4) can be applied to many areas of continuum mechanics, such as fluid mechanics, thermal physics, and electromagnetic theory, by allowing $\boldsymbol{J}$ and ρ to represent a suitable pair of physical variables.

Further Readings

On Using Stories for Teaching

R. Collins and P.J. Cooper, *The Power of Story: Teaching Through Storytelling*, 2nd ed. Scottsdale, AZ: Gorsuch Scarisbrick, 1997.

B. Lipke, *Figures, Facts, and Fables: Telling Tales in Science and Math*. Portsmouth, NH: Heinemann, 1996. (Also available as ERIC Document No. ED400200.)

On Vector Calculus

P.R. Baxandall and H. Liebeck, *Vector Calculus*. New York: Oxford Univ. Press, 1986.

H.M. Schey, *Div, Grad, Curl, and All That*. New York: Norton, 1973.

J.F. Hurley, *Intermediate Calculus*. Philadelphia, PA: Saunders, 1980.

On Conservation Law

F.B. Hildebrand, *Advanced Calculus for Applications*, 2nd ed. Englewood Cliffs, NJ: Prentice Hall, 1976.

T.P. Liu, "Hyperbolic conservation laws and related topics," in *Partial Differential Equations of Hyperbolic Type and Applications*, G. Geymonat, Ed., Singapore: World Scientific, 1987, pp. 113-139.

On Carl Friedrich Gauss

G.W. Dunnington, *Carl Friedrich Gauss, Titan of Science*. New York: Exposition Press, 1955.

W.K. Buhler, *Gauss: A Biographical Study*. New York : Springer-Verlag, 1981.

T. Hall, *Carl Friedrich Gauss, A Biography*. Cambridge, MA: MIT Press, 1970.

Being an intelligent engineer, Gar Clauss came up with a scheme to identify the property or properties where a water source or sink may be present, and, at the same time, collect sufficiently convincing data to request a court order for entry. For any property under scrutiny, he proposed to measure the rate and direction of the flow of water all around the chain-link fence surrounding that property. For each elementary section of the fence, he planned to make the following three-way determination:

1. If the direction of the flow of the water within that elementary section of the fence was parallel to the fence (which he called "tangential" flow), he would ignore that rate of flow, for it did not contribute to a net inflow or outflow of water into the property.
2. If the direction of the water flow was perpendicular (or "normal") to the fence, he would include the full rate of flow in his accounting with a negative sign if the water flow was inwards and with a positive sign if it was outwards, for it contributed to a net in- or out-flow of water from the fenced area.
3. Finally, if the direction of the water flow in that elementary section of the fence made an angle other than a right angle with the fence, he would resolve that flow into a tangential and normal component and would ignore the first but include the second in his accounting.

Gar Clauss then added the contribution from each and every elementary section of the fence to determine the net outflow of water from a given fenced lot. In this process, he was careful to make sure that he did not miss any part of the fence, so his accounting was always being carried out around a closed periphery. Since the water level was static everywhere, he argued that there should be no net in- or out-flow of water from the fence around any property in which there is no water source or sink. However, for a property in which a water source or sink is situated, there will have to be a net flux (either outflow or inflow). Thus, he could determine if a given fence circumscribed a source or a sink of water. Not only that, he could even find how big the total water flux was. This would be sufficient evidence for the court to issue a search order for the property.

Based on the collected water flow data, the city engineers were able to identify the properties that required closer onsite examination. In addition, for these properties they were able to justify court orders where needed, since the property owners were unable to contest such impeccable logic as that contained in Gar Clauss's methodology. One contested case, however, involved an unusual twist and led to an important limitation being recognized by the court. It concerned a property that was of an annular shape and that completely surrounded an environmentally sensitive zone consisting of an endangered species habitat that was not part of that property. Its owner was able to successfully argue in the court that the method of Gar Clauss did not apply to his property, since the net flows measured around the outer boundary of his property could

Gauss' Law Viewed as a Conservation Law

Gauss' law in electromagnetics is a law of physics that makes a statement about the nature of electromagnetic fields. As such, it cannot be arrived at by purely deductive means and must ultimately rely on experimental observation and inductive reasoning. The experimental basis on which Gauss' law rests is Coulomb's force law, and, indeed, the two laws are equivalent since either can be deduced from the other. Gauss' law was not discovered by Gauss, but was reformulated by him in a mathematical form that has a wider applicability (for example, invariance in the presence of materials and when observed from moving frames) than some of the other forms in which the same result can be expressed. The law relates the electric flux density $\boldsymbol{D}$ to the electric charge density ρ_c by

$$\int_S \boldsymbol{D} \cdot dS = \int_V \rho_c dV \tag{1}$$

and

$$\nabla \cdot \boldsymbol{D} = \rho_c. \tag{2}$$

These are two alternative mathematical forms of the law—an integral form and a differential form—either of which follows from the other through the use of the divergence theorem.

The continuity equation and Gauss' law are entirely different physical laws. The continuity equation involves the time variable t and arises from the conservation law. Gauss' law does not involve the time variable and arises from Coulomb's law. However, the similarity between the continuity equation and Gauss' law in (1) and (2) allows Gauss' law to be formally viewed as a conservation law. Such a viewpoint follows naturally from the concept of electric flux density $\boldsymbol{D}$ and is in the spirit of Faraday's intuitive approach (as opposed to Maxwell's mathematical approach) to understanding electromagnetic laws and phenomenon.

The essential characteristic feature of electric field that makes Gauss' law possible is the inverse-square nature of Coulomb's force law between charges. The electric field intensity, due to a point charge, decreases with distance in the same inverse square manner as the decrease in the density (or, alternatively, velocity) of a hypothetical conserved fluid that is extruded isotropically from a point source and flows radially outwards. That makes it possible to view the electric flux density $\boldsymbol{D}$ (which is proportional to the electric field $\boldsymbol{E}$) as if it were the flux of a conserved fluid, since both vary with distance in an inverse square manner. If ρ_h is the density of this hypothetical fluid, whose flux density is $\boldsymbol{D}$, then the continuity equation for this hypothetical fluid is

$$\int_S \boldsymbol{D} \cdot d\boldsymbol{S} = -\partial\left(\int_V \rho_h dV\right)\Big/\partial t. \tag{3}$$

The integral form of Gauss' law in (1) follows from (3) if the time rate of change of the density ρ_h of the hypothetical fluid is identified with the density ρ_c of the electric charge

$$\rho_c \Leftrightarrow -\partial\rho_h/\partial t. \tag{4}$$

While such an identification may appear to be entirely arbitrary, it is no more baseless than conceptualizing $\boldsymbol{D}$ as a "flux" variable that flows even under static equilibrium conditions.

have resulted from sources and sinks situated in that zone and were not definitive proof that their cause lay on his property. The ruling in that case established the legal principle that Gar Clauss could apply his methodology only to "simple regions," or unions thereof.

As the above technique turned out to be useful and became established as a standard procedure in flood surveys, it began to receive more detailed scrutiny. Soon, practicing engineers developed second-order corrections to the computed flux to account for the effect of fence wire thickness and posts. Still other engineers recognized that Clauss's method tacitly assumes the volume of water to be a conserved quantity, which would be true only if water were incompressible; hence, the need for another correction. This correction is relevant because the decompression of water produces a larger volume of it, in effect making it appear as a "source" of water (and, similarly, its compression would appear as a "sink"). With a generation term added, which turns the conservation law "volume = constant," into a hyperbolic partial differential equation, Clauss's method becomes applicable to a wider variety of engineering applications, such as analyzing shock waves in aircraft design and buoyancy in naval vessel design. As one illustrative example of the consequences of compressibility, consider the buoyancy of a large submerged object, such as a ship. People sometimes crudely articulated Archimedes' principle as asserting that buoyancy is proportional to the volume of the submerged part of the body. As a result of the increase in the density of water with increasing depth due to increasing hydrostatic pressure, a liter of water at the surface weighs less than the same volume of water at great depths. Therefore, the weight of liquid displaced by a tall submerged body is not exactly proportional to the volume of the submerged body. Indeed, Clauss's procedure (applied to pressure gradient rather than the flow rate of water) could be used to deduce that the displacement of the same volume of water at a greater depth causes larger buoyancy, thus confirming the more precisely stated Archimedes' principle that buoyancy equals the weight of actually displaced water.

As the technique was more commonly applied and its wider applicability recognized, it came to be known as Clauss's theorem. International conferences were organized to discuss the latest developments in the field. The heat transfer engineers applied Clauss's result on flow of fluids to the flow of heat. Mathematicians launched investigations into the existence, uniqueness, and bounds of the computed net flux, while physicists worried about the assumptions implicit in Clauss's theorem concerning the nature of fluids. The philosophers began exploring the philosophical implications of "measurement from a distance" on physical laws, while the legal scholars considered the need to enact new laws to protect the privacy of individuals from those who may make measurements on them from a distance. When the research sponsoring agencies got flooded with proposals for funding research projects on the subject, some wags asked if Clauss's theorem could be used to determine whether the flood of research papers had any net outflow of ideas, or was it all purely circulatory.

As the power of his theorem was recognized, Gar Clauss became well known. The city fathers were so delighted to find such a clever engineer among them that they gave him the coveted "Exemplary Service" award; the award citation read, "for deducing, from the measurement of the flux around a closed boundary, the source of the flux within." The mayor issued a proclamation that all closed surfaces would hereafter be called "Claussian Surfaces" in honor of Gar Clauss. The Revenue Department contacted him to see if perhaps he could help them estimate from a distance the flow of wealth in and out of businesses with shady reputations that wouldn't let the tax collector come near them.

The rest of this story is, unfortunately, not an inspirational tale and will therefore not be recounted here, for fear that it may disillusion the young, scuttle their idealism, and make their innocent minds cynical. It relates to the fact that the rate of outflow at a particular location, which Gar Clauss referred to as the net "divergence" of flow rate, was misunderstood by the judge, who had been a humanities major in college. The judge ruled that if the flow had diversity, then it required a broad community input and consensus and should not be subjected to Clauss's cold, one-size-fits-all logic.