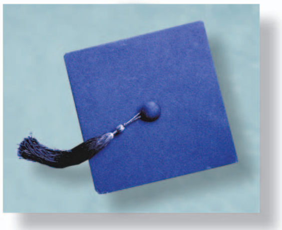


# Educator's Corner

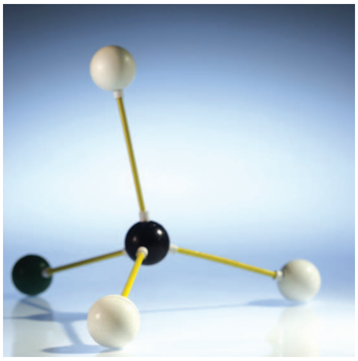
## Enhancing the Learning Value of Worked Examples

**■ Madhu S. Gupta**

There is an often-told story about a preacher who accompanied his sermon on the vices of alcoholic drinks by a demonstration. He took a few live worms and dropped one each in different bottles containing water, milk, fruit juices, and alcohol, whereupon only the worm dropped in alcohol stopped wiggling and died. The preacher expectantly looked towards the audience and asked them what lesson they learned from the demonstration. The old drunk in the back row raised his hand and offered, "If I drink alcohol, I will not have any worms in my stomach."

Although this old joke rests on the incongruent lessons that the preacher and the drunk deduced from the demonstration, it contains still another lesson—for the teachers—that is no joke. It reminds us that exposing an audience to knowledge is not enough; an instructor must also make sure that the learners take away the desired message. An instructor who assumes that the lesson to be drawn is obvious or inevitable and makes no effort to point it out explicitly engenders the risk that the students will miss the intended lesson.

Although this lesson is relevant to every type of instructional activity encountered in engineering education—lectures, laboratories, problem assignments, discussions, project work, report writing, or research work—the present column is focused on applying it solely to the worked exercises or solved problems presented as illustrative examples in the classroom (and in textbooks). Several reasons motivate such a focused attention on worked examples. First, the need to clarify what lessons a learner should draw is particularly critical for solved classroom problems because, unlike the project work and case studies that replicate the work a professional might do in real life and include the context of the work, the classroom problems are frequently idealized, context-independent exercises posed in isolation; consequently they leave the learner with a greater uncertainty and latitude in guessing the message conveyed. Second, when the problems are solved by the instructor, the path followed is under the instructor's control, and can be deliberately chosen to deliver the intended message or bring out the desired point, for example through a judicious choice of the method of solution. Third, the worked examples have a potentially large impact (compared to other types of instructional materials) on the students' problem-solving ability, since emulation is one of the most effective learning mechanisms. Finally, since the solved problems take up a significant amount of valuable classroom time to pose, formulate, solve, and discuss them, efforts to maximize the return on that time investment are advisable.



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This column is concerned primarily with ways to enhance the learning value of the instructor-directed problem-solving activity that is already occurring in the engineering classrooms. It suggests accompanying the problem-solving sessions with a classroom discussion of 1) the

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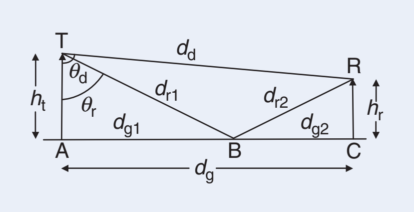
## The Sample Problem on Received Signal Strength Calculation (Based on "Two-Ray" Model of the Effect of Ground Reflection)

### Problem Statement

Consider a transmitting antenna indicated by T in the following figure, located at a height $h_t$ above ground, radiating a narrow-band RF signal at frequency $f$, which is received by another antenna indicated by R, located at a horizontal (ground) distance of $d_g$, and at a height $h_r$ above the ground. The directivity of respective antennas are $D_{tr}(\theta, \phi)$ and $D_{re}(\theta, \phi)$, and the ground is described by an average ground permittivity $\varepsilon_{gr}$, and conductivity $\sigma_{gr}$ at the signal frequency of interest. Find the strength of the received signal, and its dependence on the location of the receiving antenna (that is, $h_r$ and $d_g$).



### Lessons from the Solution of "Two-Ray Model" Problem, as Reported by Students

1) In the presence of ground reflections, the received signal power varies with range r as $r^{-4}$ rather than as $r^{-2}$ (or the path loss as $r^4$ rather than $r^2$).
2) The path loss in propagation becomes independent of wavelength in the presence of ground reflection.
3) The phase shifts introduced in signal path cause signal interference.
4) The signal power falls off very sharply beyond the distance $4h_th_r/\lambda$.
5) The motion of the receiving antenna along the line joining the two antennas will cause the received signal power to vary periodically.
6) The problem shows how multipath fading can occur.
7) Reflected signals can interfere either constructively or destructively.
8) The differential phase shift among the arriving signals is the cause for fading.
9) Small differences in the model assumptions can lead to substantially different results.

subject-specific implications of the problem and its solution, 2) the method used for solving the problem, 3) the strategies employed in selecting that problem-solving method, and 4) the constituent subproblems usable as building blocks in constructing the solution of the more complex problem. The theoretical underpinnings for each of these four suggestions, drawn from learning theory, are also briefly introduced.

## What Can Be Learned from Worked Examples

There is an old anecdote about a physician who ordered a patient to take up some form of daily physical exercise for an hour, and found no improvement after the patient had been following the regimen for two months, only to discover that the exercise the patient had taken up was horse riding, and it was the horse who was getting most of the exercise. So what can the students learn when it is the instructor who is solving the problems?

Many parts of a worked example can potentially be a source of learning. The problem statement might convey the kind of information that is typically available in practice. The given data may be indicative of the range or typical values of parameters involved. Interpretation of the problem in the learner's framework can demonstrate how to relate pieces of the problem with elements already present in the knowledge base available in the learner's memory. The problem formulation, and its transformation to a form amenable to solution, can teach the skill of problem simplification through idealization and approximation. A partitioning of the problem into a sequence of steps, and pattern matching the individual steps with canonical problems whose solution is already known to the learner, can illustrate and teach this problem-solving strategy for future use. The problem solution might explain an observation, or allow making recommendations about a course of action, or drawing conclusions about the impact of some influencing factor.

But learning from worked examples is not self-evident, and a novice can hardly be expected to absorb all the insights and nuances that a rich example makes available. In fact, experts tend to extract significantly different meanings and lessons from an example than does a novice, due to their 1) ability to relate the example to a larger and better organized database of knowledge, 2) disciplined way of analyzing and interpreting information, and 3) association of different levels of importance to individual elements of the example based on their broader perspective. Indeed the task of drawing lessons from the worked example is itself one in which skill can be developed by practice. This article presents some suggestions that can help novices benefit more from the worked examples.

To economize on the effort and overhead incurred in the following discussion below, a single worked example is used to illustrate all of the suggestions presented in this article. The selected worked example, briefly stated in "The Sample Problem on Received Signal Strength Calculation," is a problem requiring the calculation of the RF signal power received from a transmitter in the presence of both a direct ray and ground reflection. This so-called two-ray model is a classic, well-known exercise that has appeared for decades in numerous textbooks on different subjects like electromagnetic fields, wave propagation, antennas, and wireless communications. The exercise is widely used partly because it illustrates the effect of signal interference in an elementary context, and partly because it is exactly solvable in closed form with elementary mathematical manipulations, and therefore conveys the results compactly and in an easily understood form.

As an interesting aside, when this exercise was presented as a worked example in a course on RF wireless communication systems, the students were asked (prior to the more detailed discussion of the problem that followed) what lesson they learned from the example. A large variety of answers was offered, and the representative ones (after deleting those that were erroneous) are also included in "The Sample Problem on Received Signal Strength Calculation." It is apparent that, based on their own frame of reference, learners draw substantially different conclusions from the same instructional material. For all students to receive the lesson intended by the instructor, it needs to be brought to their attention.

## What It Takes to Get a Good Return on Worked Examples

The benefits of worked examples, alas, have some accompanying costs. First and foremost is the effort and time required in a judicious selection of the worked examples: clearly, superficial problems cannot be mined for deep insights. Some comments on the criteria to use in selecting the worked examples are provided at the end of this article.

A second cost is the investment an instructor will have to make in learning how to get the most out of worked examples. Discussing the thought process involved in problem solving so as to raise the metacognitive awareness of the students is not an innate ability for most instructors or subject matter experts, and is very significantly different from straight lecturing; consequently the instructors may have to step out of their existing comfort zone, and will likely have to go through a learning process themselves. This article can serve as a starting point in that process.

A third cost is the classroom time used in solving the worked examples. Although the following discussion either implicitly or explicitly refers to problem solving carried out by the instructor in the classroom, there are other alternative formats for presenting worked examples. Distribution of prepared solutions does not offer all of the same options as live problem solving, and has to be proleptic instruction, in which the instructor anticipates the learners' needs, as opposed to the live discussion with heavily verbal transmission of knowledge that permits 1) an opportunity to provide a direct, detailed explanation on demand and in real time as the solution progresses and 2) an opportunity for the learners to observe a role model engaged in professional work, with its attendant motivational benefits. Nevertheless, most of the activities suggested below are applicable even if the problem solving is not live (e.g., if written solutions are distributed, provided through a Web site, or viewed as a video recording).

Solving problems imposes additional cognitive load on a learner, thus decreasing the cognitive resources available for learning.

Sometimes the instructors are deliberately terse and sketchy in their problem solving, often in the belief that there should be some challenge left for the students to engage in an active role, instead of passively receiving the solution. While the terse approach appears to be time-efficient, it also misses an opportunity: instructors rarely stop to point out what type of problem-solving method they used, and why it was selected, because to an expert, this information is often buried at the subconscious level. As a result, the problem-solving process remains shrouded in mystery, and the students are left to discover on their own what are the lessons to be learned from a problem solution, what is crucial and what is merely peripheral in the solution, and what aspects of the solution process can be widely applied to other problems.

Admittedly, to make the solved examples more productive and enhance their learning value requires 1) an investment of additional time and effort beyond, and subsequent to, the solution of a problem, in the form of a reflective discussion which focuses on what has been accomplished and 2) assignment to the students of tasks relating to the solved problems, such as outlining the steps followed during the solution process, so as to engage the student in active learning and develop problem-solving skills. The central purpose of this article is to make some suggestions for such post-solution activities that supplement problem solving, to get the most benefit from it.

## What Makes the Worked Exercises so Valuable

The worked exercises, like the assigned ones, can serve numerous instructional purposes. The two most easily recognized and commonly articulated purposes are enhancing the learners' knowledge and skill, described below by examples; still other purposes will be discussed subsequently in this article.

- *Knowledge*: Exercises can illustrate the application of facts, concepts, and principles of the subject; exemplify the content; and convey a sense or awareness of the typically encountered designs, achievable parameter values, constraints, etc.
- *Skill*: Exercises can illustrate domain-specific algorithms and step-by-step procedures; reveal

potential sources of error, difficulty or confusion and how they can be overcome; and exhibit applicability and limitations of methodological or procedural techniques.

But, because they are solved by the instructor rather than by the student, the worked exercises differ from the assigned problems in four significant ways.

- They make both the problem as well its solution available for further exploration.
- The choice of the method of solution is governed by the instructor.
- The instructor has the ability to shift the focus from the product (that is, the problem solution) to the process (that is, problem solving).
- Not having to search for the solution of the problem significantly decreases the students' cognitive load, thereby releasing their cognitive resources for other purposes. These four differences give rise to a number of new opportunities for gaining some major learning benefits, as follows:
  - *(a) Availability of Solution*. With the problem and its solution both in evidence, their applications, implications, extension, and relevance can be brought out more thoroughly, for motivational as well as cognitive enrichment purposes.
  - *(b) Control of Problem-Solving Method*. When the problems are solved by the instructor, the method of solution becomes a controlled choice; as a result, a preferred method can be deliberately presented, so as to expose the students to efficient ways of thinking.
  - *(c) Focus on Problem Solving*. In order to learn to solve problems on their own, the learners need to acquire not only cognitive knowledge of the discipline, but also general problem-solving strategies, as discussed in an earlier article in these pages [1]. One method of enhancing the learner's problem-solving skills is by prior exposure to, and familiarity with, a rich variety of problem-solving methods, along with an awareness of their domain of utility [2]. When problems are solved by the instructor, a variety of problem-solving methods can be purposefully presented, and compared with each other, to help develop the problem-solving skills among the learners.
  - *(d) Cognitive Load Reduction*. Solving problems imposes an additional cognitive load on a learner, and therefore decreases the cognitive resources available for learning from the problems. By contrast, studying solved problems decreases the cognitive workload, and permits directing the cognitive effort to other tasks [3], [4]. As a result, the instructor can introduce problems with a higher complexity level, and help the learners develop schemata such as those that enable subject-matter experts to handle more complex problems [5].

**Cognitive resources freed due to schema formation can be used for schema creation.**

The next four sections are concerned with the ways in which an instructor can capitalize on each of these four differences, and offer some suggestions for achieving the learning benefits of worked exercises through the instructor's choice of problems, presentation of solution process, emphasis, and student activities.

## Problem Discussion for Motivation and Cognitive Enrichment

It might appear that once a problem has been solved, it has served its purpose and there is little left to discuss. On the contrary, after a problem has been solved, the learners' attention can be drawn to the purpose, context, implications, or significance of the problem, to reach instructional goals beyond the two previously mentioned goals of domain-specific knowledge and skills. The two additional instructional goals that can be served by the worked examples are as follows:

- *Motivation*: Exercises can provide a context and domain of application for the subject; capture the learners' interest; and serve as proxies for professional tasks or their subsets.
- *Perspective*: Exercises can extend and develop the content beyond previously explored boundaries; generalize or enlarge the horizons of the subject matter; and teach results of wide utility.

The learning value of worked examples can thus be significantly enhanced if the solution of a problem can be accompanied or followed by a discussion of that problem, focused not only on knowledge and skills but also on motivation and perspective. Examples of possible discussion topics which address these two additional goals are suggested in Table 1.

The value of motivating a learner in the subject or a problem at hand has long been recognized [6], [7]. In technological fields where the learners are typically self-selected and have an adequate preparation in prerequisites, the motivation for a subject matter is usually drawn from 1) its utility in practice, 2) current interest in it among the professionals, 3) its need or applicability in future learning, and 4) its relevance to the learner's personal goals. A relationship of the subject matter with current practice or state-of-the-art in the discipline, or a connection to the current needs or preferences of the industry is therefore motivational. A demonstration of practical utility can rest not only existing real-life applications but also on projected, futuristic, or possible applications. Such a discussion can be expected to boost the interest level of students who are unsure about the relevance of the classroom work to practical applications, or to current practice in the professional field. Relating the worked example to the goals, results,

**TABLE 1. Examples of discussion topics enriching the worked examples for better motivation and perspective.**

| | |
|---|---|
| 1) Educational relevance | • Why is the problem of interest? What principles or techniques does it illustrate?<br>• What is the educational role or rationale of the problem? What instructional objectives does it meet?<br>• What other topics does it relate to, or motivates the study of?<br>• What broader class of problems does the problem represent, or is an example of? |
| 2) Practical application or relevance (for motivation) | • What are the potential applications, or significance, of this result in practice?<br>• What does the solution of the problem enable? Are any systems, standards, products, processes, or techniques based on it?<br>• Do the results illustrate a trade-off, compromise, or fundamental limit encountered in practice? What technological advances have been made to beat or circumvent them?<br>• Is there a competing approach or technology presently in use, and how or why does its performance differ from that of the system in the problem? |
| 3) Relationship to state-of-the-art or current R&D (for frontier excitement) | • How do the parameter values employed or determined in the problem compare with those found in common use, or those at the state-of-the-art? What recent progress has allowed those state-of-the-art values to be reached, and in turn what has that enabled?<br>• Does any aspect of the problem relate to currently pursued R&D goals, current research literature, or unresolved problems?<br>• What problems loom on the horizon when the currently pursued goals have been reached?<br>• What options are open after the present R&D approach reaches a plateau? |
| 4) Exploration of parameter space (for developing judgment) | • What are the ranges, parametric dependences, and significance of the parameters involved in the problem?<br>• What is the expected accuracy, tolerance, or error estimates, for those parameters?<br>• What parameters are directly measurable as opposed to being deduced variables or theoretical constructs?<br>• Which of the parameters appearing in the problem are design variables, or under the designers' control, and how are they controlled in practice?<br>• Are any of the parameter values employed in the problem constrained by spatial, material, geometrical, environmental, structural, thermal, or other such constraints?<br>• Which parameters appearing in the problem are economically rather than technologically constrained, or significantly influence the cost of the system in practice? |
| 5) Implicit assumptions or constraints (for generalization and preventing misapplication) | • What assumptions are implicit in the way the system has been modeled, or the problem has been interpreted?<br>• For each assumption made in solving the problem, what are its justification, consequences, cost (that is, the restrictions or limitations caused by it), and benefits (that is, the simplifications resulting from it)?<br>• Is either the system model, or the problem-solving procedure employed, inapplicable outside some range of parameter values? What are those ranges, and what alternatives exist outside those ranges?<br>• How can the result be generalized by adding additional features to the problem, relaxing the constraints, removing the assumptions, or replacing the original problem by an entire class of problems?<br>• Do the results exemplify some general rule, or observed pattern? |
| 6) Exploration of alternatives (for application) | • Does the problem suggest an alternative to an established method, or dominant technology<br>• When can the situation or approach considered in the problem be more advantageous than its alternatives, and what are those advantages?<br>• Is there a competing technology or approach also in use, and how does its performance differ from that of the technology in the problem, and why?<br>• Can one or more elements appearing in the problem statement be substituted by an alternative which would make the results more favorable, or confer another advantage? |

or approach of some on-going research and development work in the field adds currency to the subject matter, brings the excitement of the frontier to the classroom, and does wonders for student motivation.

Developing perspective in the domain of the problem requires a discussion with a broader view point than that taken in the worked example, for example by posing what-if questions and considering the consequences of modifying the parameters or relaxing the constraints specified in the problem. The benefits of such a discussion include exposing the learners to possible generalizations of the problem, and drawing attention to the limitations and assumptions implicit in the problem and its solution, thereby warning them against the pitfall of overextending the results. It also allows the learners to widen their

## Examples of Cognitive Enrichment Topics for the Illustrative Worked Exercise on Two-Ray Model of Wave Propagation

| | |
|---|---|
| 1) Directed at domain-specific knowledge | • A list of all the variables can influence the strength of received signal.<br>• Typical values of ground permittivity and conductivity at wireless communication frequencies. |
| 2) Directed at procedural skill | • Determination of the point of reflection on the ground from similar triangles.<br>• Determination of differential path length and phase shift between the direct and the reflected rays.<br>• Determination of the reflection coefficient of the ground for a given signal wavelength, angle of incidence of signals, and polarization of radiated signals. |
| 3) Directed at motivation | • Understanding how the constructive and destructive interference can greatly influence the received signal strength.<br>• Understanding how the inverse-square power law $P_{rec} \sim r^{-2}$ in free-space following from the law of energy conservation can get modified to a $r^{-n}$ power law with the exponent n exceeding 2, which is used in the design of wireless communication systems.<br>• Understanding, from the $h_r$ and $d_g$ dependence of $P_{rec}$, how multipath fading can occur in cellular and mobile communication systems. |
| 4) Directed at perspective | • What if the distance $d_g$ is so short that the directions of the direct and reflected rays leaving the transmitting antenna, described by $\theta_d$ and $\theta_r$, are significantly different?<br>• What if precipitation makes the signal attenuation in the atmosphere sufficiently large that the path losses along the direct and reflected rays are unequal?<br>• What if the antennas are located on ocean vessels, so that the ground is replaced by ocean water?<br>• What if the earth's curvature is not negligible? |

horizons and consider more creative or imaginative alternatives.

For the sample problem on two-ray model of wave propagation, examples of possible discussion topics are shown in "Examples of Cognitive Enrichment Topics for the Illustrative Worked Exerciseon Two-Ray Model of Wave Propagation."

### Reflective Review of the Problem-Solving Method for Transferability

Although developing problem-solving skill is almost universally recognized as a valid instructional goal, there is little evidence that it is explicitly taught. The typically available problem solutions (such as those provided as solved examples in textbooks, in the instructors' solutions manuals accompanying the textbooks, and on the course Web sites) are found to be strongly focused on the subject matter content, and not on the process of problem solving. Consequently, a discussion of how a given problem is solved, and how a solution might similarly be arrived at for solving other problems of that type, is missing and left for the student to discover. One way to address those questions, and thus make the worked examples more valuable, is by following the solution of a problem with a review of the steps in the solution process, identifying the antecedents and outcomes of individual steps, with emphasis on those steps or segments of the solution that have broader applicability.

Worked examples are particularly suited for this purpose. When the problem solving is directed by a learner, the solution cobbled together may be fraught with diversions, detours, and excess baggage, and the problem-solving method helter-skelter, circuitous, unnecessarily restrictive, and inefficient [8]. Such a solution is not very conducive to understanding what has been done, let alone how it can be applied to other problems. By contrast, an instructor demonstrating problem solving can employ the intended method of solution, follow an efficient path, and highlight those elements of the solution method that are more generally useful.

Table 2 shows examples of the kind of questions that can be raised in a post-solution review of the method of solution of a worked example. They result in partitioning the problem into subtasks, each with a well-defined goal, and thus assist in teaching the following three useful lessons:

- comprehension, that is, understanding what is being accomplished at each step
- planning, that is, learning to construct a sequence of steps for solving a problem
- transferability, that is, retaining the broadly applicable steps in a generic form for reused with other problems.

**TABLE 2. Possible questions to raise in a post-solution review of the problem-solving method.**

| | |
|---|---|
| 1) Logical outline | What are the major subtasks required for solving the problem, which describe the logic, intermediate goals, or choices made in carrying out the solution of the problem? |
| 2) Sequential procedure | What ordered sequence of steps were taken to arrive at the solution, described at a sufficiently detailed level so that it could be used as a step-by-step set of directions by a newcomer for reaching the solution. |
| 3) Rationale | Why the steps in the solution are appropriate, and how would one know in advance that those steps are promising and should be attempted? |
| 4) Applicability | When or where can these steps be used for the solution of other problems (and when would they not be suitable)? |
| 5) Pattern matching | What general problem-solving method or strategy is exemplified by the solution, or each portion of it? What generic technique is it a special case of? |
| 6) Transfer opportunities | What prior problems (previously encountered by the learner) serve as models for a portion or step of the problem? What features of that portion or step should trigger a recall of the prior problems and invoke the knowledge already stored in the learner's memory? |

## Comprehension

Once a worked example has been solved, the problem solution becomes available to be retraced, dissected, and recast in suggestive forms. When a solution is retraced step-by-step, many more details and questions arise than were apparent the first time around, likely because the cognitive resources of the learners are no longer preoccupied with the search for the unknown solution. The highest beneficiaries of such a review are the weakest students who have mastered the fewest problem-solving skills, and whose cognitive resources may have been overwhelmed during the earlier solution of the problem.

## Planning

Learning a problem-solving method is particularly opportune and effective immediately following the solution of a problem that exemplifies that method. With its solution subdivided into a sequence of steps, a worked example serves as a vehicle to demonstrate problem solving as a step-by-step process, and a model to be emulated in solving other problems. A review of the subdivided parts of the worked example, each of which is individually manageable, makes the problem much more approachable, and gives the learners the confidence that they can carry out

### Examples of the Subdivision of Solution Method for the Sample Problem on Two-Ray Model of Ground Reflection

| Label | Purpose of the step |
|---|---|
| • Idealizations | Problem simplification, e.g., by assuming a flat planar ground, and lossless propagation in air. |
| • Linearity consequences | Recognition that fields due to direct and reflected received waves can differ only in direction, magnitude, and phase. |
| • Ground reflection | Calculation of the ground reflection coefficient of ground. |
| • Differential path length | Calculation of differential path length (between direct and reflected rays) by similar triangles. |
| • Phase accumulation | Determination of the net differential phase shift (due to path length and reflection) between the direct and reflected signal paths. |
| • Arrival angle | Calculation of the angle between the directions of arrival of direct and reflected waves. |
| • Relative amplitude | Calculation of the attenuation of reflected wave, relative to the direct wave, due to ground reflection. |
| • Linear superposition | Calculation of the net electric field magnitude (relative to the field due to the direct ray alone). |
| • Free-space value | Calculation of the received power due to the direct wave alone, using the Friis equation for free space. |
| • Scaling | Calculation of the net power received |

the solution on their own upon next encounter. This realization is most reassuring and useful for those students who have the fatalistic view of the problem as a monolithic whole for which they either do or do not know the solution.

### *Transferability*

Partitioning the solution of the worked example into subdivisions, each with an identifiable purpose, greatly aids in isolating and identifying those subdivisions that are broadly applicable and are therefore worthy of retention in the long-term memory. The ease with which this information can be transferred to other problems can be further increased if the steps taken in solving the problem are described in a generic or abstract manner, without reference to the specifics of the problem [5]; to do so will typically require additional effort, since the solved problems will usually be presented in a subject-specific context. The broadly applicable steps in the solution can be found in many stages of problem solving, including the interpretation of the problem, idealization and modeling of the situation, formulation in a solvable form, and pattern matching with known canonical solutions [1]. Thus the post-solution review of a worked example makes the information to be retained apparent to the learner, manageable in extent, and better organized for retention; with judicious selection of problems, it also enables extracting from them efficient schemata suitable for retention in the learner's long-term memory, retrieval, and reuse on a subsequent occasion [5].

**The goal of learning is to prepare for the novel rather than clone the venerable.**

Several student activities can be employed to keep the learners engaged and help them adopt the problem-solving method. Given a problem solution, the learners can be tasked with identifying the major subgoals and the how they are concatenated in the progression towards solution. To provide learning incentive, feedback, and reinforcement, a worked example can be followed by a problem assignment which requires the use of the lessons learned in the worked example. The students can also be asked to conceive or search other problems amenable to solution by the discussed method.

For the sample problem on calculating the received signal strength using the two-ray model with ground-reflection, "Examples of Subdivision of Solution Method for the Sample Problem on Two-Ray Model of Ground

**Examples of Different Knowledge Types Defined in Bloom's Taxonomy for the Sample Problem on Two-Ray Model of Ground Reflection.**

| **Knowledge type** | **Example from the two-ray problem** |
|---|---|
| **Cognitive:** | |
| Factual | Definition of the reflection coefficient for an electromagnetic wave incident at a reference plane |
| Conceptual | The dependence of the reflected and transmitted wave amplitudes at the interface between two media, on the media parameters ($\varepsilon$, $\mu$), the signal frequency, and the angle of incidence of the electromagnetic waves |
| Procedural | Calculation of the reflection coefficient of electromagnetic waves for oblique incidence at the planar interface between two different electromagnetic media<br>(Reflection coefficient at a planar interface usable only for interfaces with dimensions much larger than a wavelength, with effective parameters ($\varepsilon$, $\mu$) assigned to a medium provided inhomogeneities in it are limited over distance scales much smaller than a wavelength). |
| **Metacognitive:** | |
| About self (learner's strengths, weaknesses, tendencies, and preferences in the knowledge domain) | Preference for conceptualizing the wave reflection in terms of a transfer function $\Gamma$ (here found by matching field components at the boundary between media), rather than in terms of traveling waves (propagating on transmission lines of different characteristic impedances modeling the different media of propagation) in forward and reverse directions. |
| About problem solving (methods and strategies; their applicability and utility) | *The perturbation strategy*: If a problem can be solved, or its solution is known, for a simpler special case (here, the power received due to direct ray alone is solvable through Friis equation), then that solution can be treated as a "baseline," and the original problem is then reduced to that of finding the "correction factor" (here the ratio of the total field strength to that due to the direct ray alone) to be applied to the baseline solution. |

Reflection" shows the outline of the logical steps in the solution of the problem, and one possible sequential ordering. The subdivisions of the solution process can be explored further to clarify such aspects as rationale for the problem-solving method, which is based on the linear superposition principle, and its applicability, which is ensured by the linearity of the media encountered along the path of propagation.

## Discussion of Problem-Solving Methods and Strategies for Enhancing Metacognition

The learning resulting from worked examples can occur not only at the disciplinary (or subject-specific) level but also at the metacognitive level which transcends disciplines, and is briefly described below. Understanding the following discussion will be helped by some familiarity with the revised Bloom's taxonomy of educational objectives in the cognitive domain, which has been described earlier in these pages [2]. Classroom problem solving, as typically carried out, is aimed at developing the learners' competence in subject matter content and methodological skills. In terms of Bloom's taxonomy of educational objectives in the cognitive domain [2], these goals relate to the factual, conceptual, and procedural knowledge of a subject matter. In addition, (the revised) Bloom's taxonomy recognizes a fourth level of cognitive dimension—called metacognitive knowledge—which transcends the three discipline-specific components of knowledge and is essential to developing higher-level cognitive skills. The classification of the knowledge of a discipline as cognitive and metacognitive, and its further subdivisions, taken from the taxonomy, are reproduced in "Examples of Different Knowledge Types Defined in Bloom's Taxonomy for the Sample Problem on Two-Ray Model of Ground Reflection."

It is difficult to find examples in the current practice of engineering education where the instruction deliberately and explicitly addresses instructional goals at the broader metacognitive level. One reason might be many instructors' reluctance to spend valuable classroom time on developing some generic problem-solving strategies, believing them to be outside the scope of a discipline-based course, possibly in part because the instructors themselves carry out problem solving in their own disciplines like experts without explicitly thinking about the generic problem-solving methods and strategies they might be using. Worked examples offer the instructor an opportunity for an explicit focus on this goal.

**Instructors must make sure that the learners take away the desired message.**

### What Is Metacognitive Knowledge

Metacognitive knowledge is knowledge about knowledge [9], [10], and includes, in addition to learner's self-awareness, an awareness of the general learning and problem-solving strategies relevant to the discipline, and their applicability in that domain of knowledge. To clarify the distinction, "Examples of Different Knowledge Types Defined in Bloom's Taxonomy for the Sample Problem on Two-Ray Model of Ground Reflection" shows examples of each knowledge type, taken from the two-ray problem stated earlier. The knowledge at the metacognitive level transcends the specifics of subject matter and is broadly useful in thinking about, strategizing the approach to, and carrying out the solution of, a problem. Examples of metacognitive strategies include the ability to plan, monitor progress, correct errors, and utilize the feedback from observations to redirect effort.

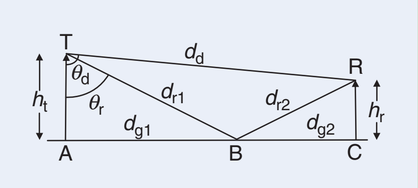
### Why Is Metacognitive Knowledge Useful

Metacognitive knowledge enables individuals to make deliberate, informed choices about their course of action, reflect purposefully and systematically about their performance, and use this information to modify or redirect their future performance and thinking. In the absence of metacognitive awareness, the learners are likely to disregard the feedback information that one gets from the empirical evidence in the course of problem solving, instead of benefitting from it by using it to enhance or rectify their metacognitive knowledge, thereby missing the growth opportunity. As discussed in an earlier article in these pages [1], the principal barrier in problem solving is making the transition from a problem that has already been interpreted and understood in a learner's framework, to the construction of a procedure for its solution. This involves making many decisions and judicious choices among alternatives, which requires both the ability to make comparisons, as well as the personal traits of confidence, persistence, and willingness to be adaptive. Hence the need for metacognitive knowledge.

### Increasing Metacognitive Awareness Through Worked Examples

A knowledge of the personal traits as well as of problem solving methods can be developed through observing and attempting problem-solving, for which worked problems provide an opportunity. It is not such an easy task for students to learn on their own how to compare alternative approaches to the solution of a problem with respect to the resulting complexity and effort required, or contrast their effectiveness, and articulate the way in which the solution methods differ. With worked examples, a variety of problem-solving methods can be deliberately presented, thus exposing the students to multiple approaches, alternatives, and strategies, while modeling the expert behavior for the students to emulate. The worked examples serve as a vehicle for the post-solution discussion to raise the metacognitive awareness, by explicitly pointing out the methods employed for problem solution; the rationale for selecting each; the kind of information required for

## Complexity of the Two-Ray Model of Ground Reflection



### Given Parameters

- Transmitter related: signal power $P_{tr}$ and frequency $f$ (or wavelength $\lambda$)
- Transmitting antenna related: antenna directivity $G_{tr}(\theta, \phi)$, and efficiency $\eta_{rad}$
- Receiving related: ground permittivity $\varepsilon_{gr}$, conductivity $\sigma_{gr}$
- Ground related: ground permittivity $\varepsilon_{gr}$, conductivity $\sigma_{gr}$
- Geometrical parameters: heights $h_t$ and $h_r$ of antenna, their ground separation $d_g = d_{g1} + d_{g2}$

### Results of Interest

- Signal power received by the receiver, $P_{rec}$ (or the net path loss, $L_{path}$)
- Effect of receiver movement (horizontally or in height) on received signal power, $P_{rec}(d_g)$ and $P_{rec}(h_r)$

### Sources of Complexity

- Large number of influencing variables
- Some parameters influence the results in multiple ways; e.g., antenna separation influences total pathloss through path length, direction of wave propagation with respect to main lobe (and hence transmitting antenna gain), as well as the angle of incidence on ground (and hence the reflection coefficient)

### Strategies for Reducing Problem Complexity

- *Idealization* (e.g., flat earth; unobstructed line-of-sight path; homogeneous ground having effective $\varepsilon$ and $\mu$ values; and smooth ground allowing the use of the reflection coefficient for planar interfaces and ignoring scattering)
- *Postponement*, or temporary elimination of peripheral details (e.g., treating the propagation through air as lossless, and antennas as isotropic) which can be accounted for later by introducing corrections.
- *Assimilation* or choice of working parameters which lump together multiple effects or features (e.g., use of antenna gain rather than directivity to avoid having to account for antenna losses; or of power radiated by antenna rather than that generated by the transmitter because it already incorporates transmitter losses; or of EIRP instead of transmitted power as it encompasses transmitter gain)
- *Segmentation* through the use of intermediate variables which either decouple the problem into two sequential problems, or subsume some detail that can be worked out separately and independently (e.g., employing the ground reflection coefficient $\Gamma(f, \varepsilon, \sigma, \theta_i)$ which separates the determination of incidence angle from the geometrical parameters and the calculation of signal power from the reflection coefficient)

and resulting from using the method; limitations of individual methods; conditions under which each is useful; the range of applicability, utility, generality or versatility of each method; the manner in which the instructor determined what the next step ought to be; and their use for constructing the solution. Such a process puts into evidence the subtle choices and decisions that an expert problem solver makes along the way, many of them unconsciously, and answers the question "where do we go from here" at each step by revealing how at each branch point the instructor determined which path should be taken or abandoned.

Although the solutions to the worked examples are provided to the students, Table 3 lists a number of possible student activities that can serve as (or be added to) problem statements for worked examples, to engage the student in active learning and to promote metacognition among them.

## Adapting Complex Problems for Student Understanding

Non-trivial engineering problems often pose a complex cognitive task. The complexity can stem for a number of sources such as the following:

- a large number of interacting elements or variables involved in the problem, with some possibly influencing or depending on others
- multiple ways in which some parameters enter into or impact the result
- dependence of available choices (such as course of action or alternative approximation) on some feature or parameter value
- the requirement that the solution must satisfy multiple simultaneous constraints or equations, or be self-consistent.

It is common for instructors to select the more complex problems on a topic for solving in the classroom, recognizing that such problems would overwhelm the students' cognitive resources, and therefore would appear to be inordinately difficult for the students to solve on their own. This ancient and instinctive choice in fact has a perfectly valid rationale in modern learning theory. There is a very sizable body of experimental work in learning science that has led to the formulation of a cohesive theory, called "cognitive load theory" (CLT), which concludes that a complex problem can indeed cause a cognitive overload, and prevent learning; references [3] and [4] are reviews of that work. The following is a very brief summary of the basic tenets of this theory, and its relevance to using worked

**TABLE 3. Tasks related to problem-solving strategy for promoting metacognition.**

- Identify the principal components or subsections of the system described in the problem, and the variables of interest in each.
- List the parameters involved in the problem, and classify them as known (that is, given), required (that is, unknown and desired), intermediate (unknown and not required), and constants.
- Given the detailed solution, outline the grand plan, and divide it into major steps in the solution.
- Break down the problem into a cascade of simpler problems, solved sequentially, and label subproblems with descriptive names that recall their most significant feature.
- Subdivide the problem into subproblems, each with a distinguishable goal, and state the relationship of each subgoal to the overall problem.
- What are the limitations to the applicability or utility of the method employed?
- Why was the employed method chosen to solve the problem? What clues in the problem statement portend the use of the method employed?
- Carry out a qualitative solution, by listing the sequential steps, identifying the information generated at each step, all the antecedents required for generating it, and the use of that information at a later step.
- Propose two different methods of solving the problem, compare their applicability, and identify how they differ from each other.
- Construct an alternative method that could be used for solving the problem, and the resulting advantages and disadvantages.
- For each assumption, approximation, or simplification used in the problem solution, show where and why it was needed, and either provide a justification / rationale for it, or determine its consequences, that is, how the results are restricted in their applicability as a result.
- Find some limiting, special, or asymptotic cases in which the results of the problem can be checked against known or expected behavior.

examples as an avenue for introducing complex problems to learners.

### Characterizing Cognitive Complexity

Depending on the purpose, the level of complexity or difficulty of a task or problem has been variously quantified and characterized in terms of the length of the chain of serial subtasks into which the overall task can be decomposed; the number of distinct operations performed; the amount of time required in performance; the overhead involved in managing the flow of information among the subparts of the task that must be separately carried out in parallel or sequentially; or the number of steps required to unambiguously specify a procedure or algorithm for carrying it out (that is, its algorithmic length). For human cognitive tasks, perhaps the most useful measure of the cognitive load of a problem or task is one that recognizes the primary limitation and bottleneck of the human knowledge processing [3], [4], namely the capacity of human working (short-term) memory, which is limited to only about 7 ± 2 elements. In CLT, the cognitive load is measured in terms of the largest number of interacting items of information that must be simultaneously processed in the working (short-term) memory of the learner due to their mutual interaction, in carrying out the task (or the most complex of the subtasks comprising the task). An overload occurs when this number exceeds approximately 7 (±2), which is the limit of human short-term memory capacity.

**Learning from worked examples is not self-evident.**

### Schema or Chunk Formation

The experts and professionals who have mastered a domain of knowledge get around the limitation of limited working memory in several ways. First, with increasing proficiency, some of the elements of information that must be processed simultaneously become part of a single module, and get subsumed into a schema (sometimes called "chunk"), which can be processed in the working memory as a single element, thereby reducing the overall cognitive load. Second, the schemata are hierarchically organized, so that a higher-order schema can incorporate several lower level ones, thereby decreasing the total number of schemata that must simultaneously occupy the working memory. Finally, with sufficient practice, the information that is present in the long-term memory and organized in a schema can be employed without conscious awareness, obviating the need for taking up any of the precious short-term memory at all, and resulting in still further reduction in the load on working memory; such schema are said to have been "automated." Indeed, expertise in a knowledge domain can be defined as the degree of modularization or schema acquisition, and their automation, along with the degree to which those schema contain the information relevant to the problems typical in that domain. Therefore, the complexity of a task is more generally specified by both the number of elements that must be simultaneously processed, and the order of the schema involved in the processing.

**TABLE 4. Considerations in selecting problems to be used as worked exercises.**

| | |
|---|---|
| • Knowledge domain | What is the relationship of the problem to the knowledge and skills in the subject matter or topic under discussion? |
| • Anticipatory value | Does the problem lead to a result or insight that is subsequently required in upcoming work or study? |
| • Instructional objective | What educational objectives are addressed by the problem, and at what level (in Bloom's taxonomy)? |
| • Motivational value | Does the problem relate to any practical applications, new or recent developments, or topics of current or historical significance? |
| • Generalizability | Whether the problem illustrates, or can be generalized for, the solution of an entire class of problems? |
| • Canonical nature | Whether the problem is encountered frequently, or in a variety of situations, either by itself, or embedded within a larger problem? |
| • Schema formation | Can the problem help the learner construct a mental framework or schema useful for integrating subsequent knowledge acquisition? |
| • Complexity | Does the problem illustrate the interaction of multiple factors, or a particularly involved, nuanced or difficult procedure? |

## Cognitive Load

When the goal of learning is to attain expertise in a knowledge domain, it is clear that the instruction should: 1) facilitate modularization or schema acquisition; 2) aid in the construction of higher-level schema incorporating other lower-level schemata; and 3) automate the acquired schema. One of the purposes of having students solve problems is to allow them an opportunity to form the schema or "chunks" at least for commonly-occurring rudimentary steps, which is a very valuable instructional goal. However, the task of problem solving itself imposes an additional cognitive load, and therefore decreases the cognitive resources available for schema formation. Moreover, as cognitive resources are freed due to schema formation, they can be used for schema creation, a highly desirable occurrence but one that also imposes additional cognitive load, called "germane cognitive load" [11].

## Complexity Induced Learning Diminution

Since a task requiring problem solving imposes additional cognitive load on the learner due to the necessity of having to solve the problem, the learners are limited in the complexity of problem-solving tasks that they can undertake. Given a task of high complexity, a learner can fail to learn from it for a variety of reasons:

- *Fatigue.* The learner may need to spend so much time that there is weariness, discouragement, or loss of motivation and interest.
- *Amorphousness.* The learner may to reach the solution of the problem but not be able to extract from the detailed solution a general strategy that could be deployed in general.
- *Distraction.* The problem may contain tangents or diversions that are not central to the theme or message of the problem.
- *Masking.* The problem may contain peripheral issues that may mask the central message of the problem.

### Examples of Cognitive Load Reduction Through Formation of Frequently Occurring Schemata (or Chunks) for the Sample Problem on Two-Ray Model of Ground Reflection

Known, frequently occurring canonical problems (or schemas) embedded within the larger problem:

- The free-space propagation problem ("Friis equation")
- Superposition of a signal and its phase-shifted replica

Schema 1
Power of two superimposed harmonic signals with phase difference

$$|E_{dir}+E_{ref}|^2 = |E_{dir}|^2\left[1+\frac{|E_{ref}|}{|E_{dir}|}e^{j\Delta\phi}\right]^2$$

Schema 2
Power received for Free-space propagation of signals (Friis Equation)

$$P_{rec} = P_{tr}G_{tr}G_{rec}\left(\frac{\lambda}{4\pi d}\right)^2$$

Higher-order schema
(Incorporates both schemata 1 and 2).

$$P_{rec} = P_{tr}G_{tr}G_{rec}\left(\frac{\lambda}{4\pi d}\right)^2\left[1+\frac{|E_{ref}|}{|E_{dir}|}e^{j\Delta\phi}\right]^2$$

- *Misclassification*. The learner may spend a significant fraction (or even a majority) of the time solving one aspect of the problem, and even if that aspect is not the central theme of the problem, would be inclined to view that aspect as the significant feature of the problem, and would classify the problem with that aspect as the primary identifier, thus hindering its transfer to other situations.

The reduction in cognitive workload of the learners, resulting from following a worked example instead of having to solve it themselves, provides an opportunity to introduce problems of higher complexity [12]. The worked examples thus represent a way to assist the students in successfully learning to perform tasks of higher complexity level. More importantly, they can help create schema to reduce the apparent complexity of a similar problem in the future.

To illustrate how a complex problem can be adapted for student understanding, consider the example of the "two-ray model" of ground reflection, presented earlier in "The Sample Problem on Received Signal Strength Calculation." The complexity of the problem arises from the large number of impacting variables, and from their interaction through jointly influencing the features that govern the results, as indicated in "Complexity of the Two-Ray Model of Ground Reflection." An efficient problem-solving method therefore manages the complexity of the problem by using the well-known strategies for complexity reduction; examples of such strategies are included in "Complexity of the Two-Ray Model of Ground Reflection."

To assist the learner in forming mental schemata of enduring value, it is necessary to identify the commonly occurring subproblems that might be lurking within the larger complex problem at hand. A prior familiarity with such canonical problems permits the learner to rapidly partition the complex problem into subdivisions whose solutions are already known and can be stitched together; this is the *modus operandi* of an experts, who operate with the knowledge of a larger set of more complex canonical problems. "Examples of Cognitive Load Reduction Through Formation of Frequently Occurring Schemata (or Chunks) for the Sample Problem on Two-Ray Model of Ground Reflection" shows examples of some schemata which, if deployed, greatly reduce the complexity of the posed problem. Therefore a particularly effective instructional strategy is to pose and solve the subproblems prior to taking up the complex problem, to help the learners both form the schemata and then deploy them. Indeed, the entire complex problem posed in "Complexity of the Two-Ray Model of Ground Reflection" can be treated as a super-schema, that can be used for solving still more complex problems involving mobile transmitters/receivers, and multipath fading.

## Selection and Use of Worked Examples

Randomly selected problems, or those created solely to provide busy work for students, are unlikely to be a valuable source of learning. Reaching significant learning goals requires that the problems address the cognitive abilities from various levels of the hierarchy of abilities in Bloom's taxonomy, particularly at the higher levels [2]. In addition, worked examples that are to be used to teach problem solving must illustrate many different problem-solving methods and strategies. Thus the selection of worked-out exercises requires careful attention to several concurrent considerations, some of which are summarized in Table 4, although a high worthiness in just one of the considerations may be sufficient to justify using a problem as an example.

Confronted with the one of the fundamental tenets of learning theory, that every learner must construct his or her own knowledge for himself or herself, instructors sometimes lament the fact that knowledge cannot be transferred directly from the instructor's memory to the learner's memory. The solution of a worked example, followed by a discussion of the applications, methods, strategies, and schemata illustrated by it, is perhaps as close as we can come to accomplishing that direct transfer. But the goal of learning is to prepare for the novel rather than clone the venerable. Therefore, like any other instructional material, a worked example and its associated discussion must ultimately be judged by their success in enabling the learners to tackle new situations outside the classroom.

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