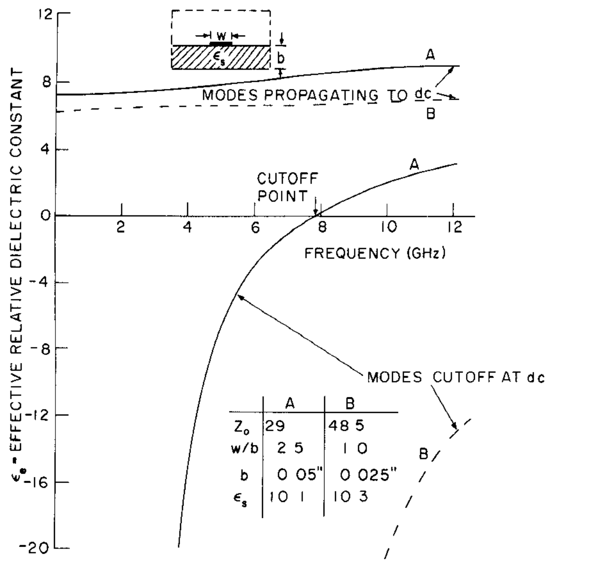

	A	B
Z_0	29	48 5
w/b	2 5	1 0
b	0 05"	0 025"
ϵ_s	10 1	10 3

Fig. 3. DC propagating and cutoff modes in microstrip from coupled-line model.

$$\omega_0^2 = \frac{\mathcal{K}^2 v_0^2}{\bar{\epsilon}(1 - k^2)} \tag{11}$$

where ω_0 is the cutoff frequency of the mode which does not propagate at dc. This relation probably should be used with some caution since the parameters of the model are based solely on the functional mode, but (11) may be useful when considering the high-frequency limitations of microstrip.

Acknowledgment

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A Small-Signal and Noise Equivalent Circuit for IMPATT Diodes

MADHU-SUDAN GUPTA

***Abstract*—A frequency-independent small-signal equivalent circuit for an IMPATT diode is proposed. It incorporates five circuit elements, including a negative resistance, and is valid over an octave range of frequency. With the addition of two white noise sources it also serves as a noise equivalent circuit.**

Introduction

An equivalent circuit of an electron device is a linear network having the same terminal properties as the device. Equivalent-circuit representations have been established for many electron devices because they facilitate the study of effects related to the external circuit. Frequency-independent equivalent circuits are particularly useful because they permit the use of simple circuit analysis techniques and aid in the study of the frequency variation of device performance.

For a nonlinear two-terminal negative-resistance device like an IMPATT diode, a linear equivalent circuit can be found for the small-signal (linearized) behavior of the device and for a limited frequency range of validity. The purpose of this short paper is to present a small-signal equivalent circuit and a noise equivalent circuit for IMPATT diodes. These equivalent circuits approximate only the terminal behavior of the diode;[1] no physical significance is attached to the circuit elements.

No frequency-independent lumped-noise equivalent circuit for IMPATT diodes has been reported so far. Haus *et al.* [1] have found a noise model for IMPATT diodes in the form of a transmission line with distributed noise sources which is not as convenient as a lumped-noise equivalent circuit. Johnson and Robinson [2] have, on the other hand, used a frequency-dependent model formed by separating the IMPATT-diode impedance into avalanche-region and drift-region impedances and connecting a noise source with the avalanche-region impedance.

A suitable small-signal equivalent circuit is also not available in the literature. The results of most theoretical calculations [3]–[6] and experimental measurements [7]–[9] of the small-signal impedance of IMPATT diodes have been expressed as a frequency-dependent admittance. Steinbrecher and Peterson [10] have proposed a frequency-independent small-signal model which is accurate only in the limit of low frequency ($\omega\tau_d \lesssim \pi/4$, where τ_d is the drift-region transit time) and predicts a diode negative conductance whose magnitude increases monotonically with frequency to an asymptotic value. Typical X-band diodes, however, have a maximum negative conductance at a frequency where $\omega\tau_d \approx 0.8\pi$, or higher for higher bias current [11], above which the magnitude of conductance decreases with increasing frequency; the model in [10] is, therefore, not suitable in the most useful frequency range of the diodes. Hulin *et al.*, [12] have also reported a circuit representation for the avalanche region of IMPATT diodes.

A frequency-independent small-signal equivalent circuit for an avalanche transit-time diode operated in the IMPATT mode, incorporating a negative resistance as the active element, is presented here. A noise equivalent circuit can also be derived from this small-signal model by incorporating two noise current sources in the circuit. Both sources are constant and frequency independent and are fully correlated with each other.

The equivalent circuit of the package in which the IMPATT diode is mounted is usually considered an integral part of the diode. In experimental evaluation and use of the diode equivalent circuit presented here, the diode package will have to be accounted for [7], [8]. An equivalent circuit for the package (which depends upon the method of mounting the package in a cavity) should, therefore, be added to the diode equivalent circuit.

Method of Determination

The two basic methods for determining the equivalent circuit for a given diode are the following.

From $Y_D(\omega)$ and $\overline{e_n^2}(\omega)$

For accurate modeling, the equivalent circuit is evaluated using the small-signal diode admittance $Y_D(\omega)$ and the mean-square open-circuit noise voltage per unit bandwidth $\overline{e_n^2}(\omega)$ at the diode terminals. For a given diode, these may be determined either directly by experimental measurement (and de-imbedding the diode from its circuit [10]) or indirectly, by first determining the diode structure (i.e., doping profile by CV measurement) and then carrying out theroetical calculations using a model such as the small-signal analysis of Gummel and Blue [6] which can be used for calculating both $Y_D(\omega)$ and $\overline{e_n^2}(\omega)$ numerically. In either case, the values of network elements in

[1] A small-signal equivalent circuit of the device will be defined as one having approximately the same terminal impedance as the small-signal device impedance, and a noise equivalent circuit of the device as one for which both the impedance and the open-circuit noise voltage are close to those for the device. Obviously, a noise equivalent circuit will also serve as a small-signal equivalent circuit upon omitting the noise sources from it.

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the equivalent circuits can now be evaluated by selecting them for a best fit to the frequency dependence of the small-signal diode admittance $Y_D(\omega)$ and mean-square terminal noise voltage $\overline{e_n^2}(\omega)$.

From Read-Model Approximation

When only an approximate calculation is desired, the element values may also be determined using the Read-model approximation [3] of IMPATT diodes. The approximate circuit element values so obtained may be adequate when an equivalent circuit is required for a batch of diodes, and the diode-to-diode variation makes it unnecessary to obtain the elements optimally by numerical best fit. The approximate element values can also serve as a starting guess in some numerical curve-fitting procedure for obtaining a more accurate equivalent circuit.

For such rough estimation purposes, the Read model of Gilden and Hines is adequate, for which both the small-signal admittance [4] and the noise voltage [13] have been calculated. The small-signal impedance is given by[2]

$$Z_D(\omega) = R_s + \frac{l_d^2}{v_s \epsilon A}\left\{\frac{1}{1-\frac{\omega^2}{\omega_a^2}}\right\}\frac{1-\cos\theta}{\theta^2} + \frac{1}{j\omega C_d}\left\{1 - \frac{\sin\theta}{\theta} + \frac{\frac{\sin\theta}{\theta} + \frac{l_a}{l_d}}{1 - \frac{\omega_a^2}{\omega^2}}\right\} \tag{1}$$

and the open-circuit mean-square noise voltage per unit bandwidth [14] is given by

$$\overline{e_n^2}(\omega) = a^2 \frac{V_B^2}{I_{dc}} \cdot \frac{1}{\left[1 - \left(\frac{\omega}{\omega_a}\right)^2\right]^2} \tag{2}$$

where

- ω_a avalanche frequency;
- R_s series resistance of the diode;
- v_s saturated velocity of carriers in semiconductor;
- ϵ permittivity of the semiconductor;
- A area of cross section of the diode;
- C_d drift-region capacitance of diode $(=\epsilon A/l_d)$;
- l_a length of avalanche region;
- l_d length of drift region;
- τ_d transit time of drift region $(=l_d/v_s)$;
- θ $=\omega\tau_d$;
- V_B breakdown voltage;
- I_{dc} dc bias current;
- a^2 constant defined in [14] $(=3.3\times10^{-20}$ C for silicon (Si) diodes).

The more general results of Gummel and Blue [6] will also reduce to these equations upon making some additional assumptions.

Note that (1) can be rewritten so that it involves only four[3] independent Read-diode parameters: τ_d, C_d, l_a/l_d, and ω_a. The first three can be found by doping profile measurement; in particular, the ratio l_a/l_d may be evaluated[4] from available plots [11]. The fourth depends, in addition, upon the operating point of the diode (I_{dc}) and can be measured or estimated [7]. One further structure-dependent parameter V_B is included in (2), and it can be directly measured. Thus a knowledge of the structural dimensions, material properties, and operating point of the diode can be used for evaluating $Y_D(\omega)$ and $\overline{e_n^2}(\omega)$, and hence the equivalent circuit, using the Read-diode model. Expressions will be given below for an approximate evaluation of the equivalent-circuit elements in terms of these parameters, thereby circumventing the need for curve-fitting.

Small-Signal Equivalent Circuit

In general, the equivalent circuits are not unique, and many of them can usually be synthesized for a given device. The number of lumped elements used in the equivalent circuit requires a compromise between complexity and difficulty of evaluation on one hand, and the frequency range of validity on the other. The smallest number of network elements required in the equivalent circuit for a reasonable approximation to the two-terminal behavior of an IMPATT diode is five. The present circuit incorporates five network elements and can be used over an octave of frequency range centered at the frequency of maximum negative conductance, which is approximately the frequency range over which practical use is made of the diode as a negative-resistance device.

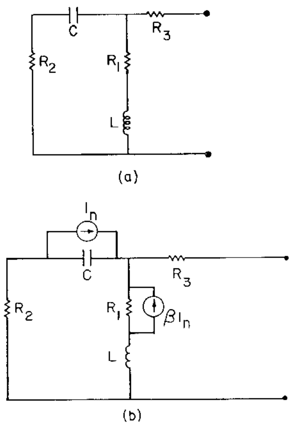

Fig. 1. Frequency-independent equivalent circuit for IMPATT diodes. R_2 is a negative resistance. (a) Small-signal equivalent circuit. (b) Noise equivalent circuit.

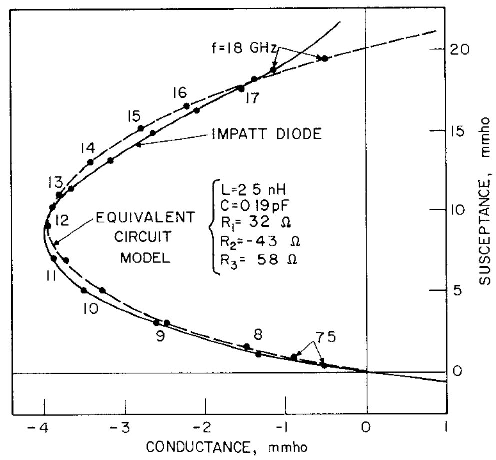

Fig. 2. Small-signal admittance plot of a Si Read diode and the admittance of the equivalent circuit.

The IMPATT diode can be represented by the circuit shown in Fig. 1(a) where R_2 is a negative resistance and all the elements are frequency independent. Fig. 2 shows the frequency dependence of the admittance of the equivalent circuit. The solid curve represents the admittance of a Read diode for which an equivalent circuit is to be found. The broken curve is the admittance of the equivalent circuit with the values of the circuit elements listed in Fig. 2. It is evident from the figure that the equivalent-circuit representation is fairly accurate for the frequency range 8–16 GHz. The solid curve of Fig. 2 was determined by a small-signal analysis of a Si IMPATT diode, and the circuit element values were numerically determined to match the diode admittance curve. The diode series resistance R_s has not been included in the plots of Fig. 2; it may be added directly to the resistance R_3 of the equivalent circuit.

When the Read model is used for the IMPATT diode, the circuit element values may be calculated as follows. The impedance of the two-terminal network of Fig. 1(a) is

$$Z_{eq}(\omega) = \frac{(R_1+R_3) + j\omega(L + R_1R_2C + R_1R_3C + R_2R_3C) - \omega^2 LC(R_2+R_3)}{1 + j\omega C(R_1+R_2) - \omega^2 LC}. \tag{3}$$

The Read-diode impedance given by (1) can be approximated in the neighborhood of $\theta=\pi$ rad by using $(1-\cos\theta)/(\theta^2/2)\approx 1-0.06\theta^2$

[2] An error of a factor of 2 in [4] has been corrected.

[3] Although the Read model has four degrees of freedom, a frequency-independent equivalent circuit may require more than four network elements if it is to be valid over a wide frequency range.

[4] While the analysis leading to (1) assumes that l_a/l_d is small compared to unity, the effect of finite avalanche-region width is nevertheless included to a first order of approximation.

and $(\sin\theta/\theta) \approx 1 - 0.1\theta^2$ (and excluding R_s which can later be added to R_3). The impedance is given by

$$Z_D(\omega) = \frac{\dfrac{\omega_a^2\tau_d}{2C_d}(1 - 0.06\omega^2\tau_d^2) + \dfrac{j\omega}{C_d}\left(1 - 0.1\omega_a^2\tau_d^2 + \dfrac{l_a}{l_d}\right)}{\omega_a^2 - \omega^2}. \tag{4}$$

Upon comparing (3) and (4), the circuit element values are given in terms of diode parameters by the following equations:

$$R_3 = \frac{\tau_d}{4C_d}(1 + 0.06\omega_a^2\tau_d^2) \tag{5}$$

$$R_1 = R_3 - 0.03\frac{\omega_a^2\tau_d^3}{C_d} \tag{6}$$

$$R_2 = -R_1 \tag{7}$$

$$L = \frac{1 - 0.1\omega_a^2\tau_d^2 + l_a/l_d}{2\omega_a^2 C_d} + \sqrt{\frac{R_1^2}{\omega_a^2} + \left(\frac{1 - 0.1\omega_a^2\tau_d^2 + l_a/l_d}{2\omega_a^2 C_d}\right)^2} \tag{8}$$

$$C = \frac{1}{\omega_a^2 L}. \tag{9}$$

An approximate small-signal equivalent circuit is thus determined by (5)–(9).

The equivalent circuit determined by either of the above methods is bias-current dependent because both $Y_D(\omega)$ and ω_a depend upon I_{dc}.

Noise Equivalent Circuit

The noise generated in an IMPATT diode can be modeled by incorporating noise sources in the small-signal equivalent circuit of Fig. 1(a). One of the several possible noise models is shown in Fig. 1(b) where I_n and βI_n are the rms noise currents per unit bandwidth of two white (frequency-independent) noise current generators that are completely correlated with each other. The value of the noise equivalent circuit lies in this feature of the frequency independence of noise sources [15]. The open circuit rms noise voltage across the circuit terminals in a 1-Hz bandwidth is, therefore, given by

$$\sqrt{\overline{e_{eq}^2}} = I_n \left| \frac{R_1 + j\omega L + \beta(R_1 + j\omega C R_1 R_2)}{1 + j\omega C(R_1 + R_2 + j\omega L)} \right|. \tag{10}$$

The constants I_n and β in the noise equivalent circuit of Fig. 1(b) can be so chosen that the noise voltage $\overline{e_{eq}^2}(\omega)$ given by (10) matches the diode noise voltage $\overline{e_n^2}(\omega)$ and its frequency variation as determined by measurement [14] or calculation [6] over the frequency range of interest.

In some applications of IMPATT diodes, the low-frequency (video) noise generated by the diode may play a significant role and can also be calculated from the equivalent circuit presented. In the limit of low frequency, (10) takes the form

$$(\sqrt{\overline{e_{eq}^2}})_{lf} = (\beta + 1)R_1 I_n. \tag{11}$$

If the validity of the noise model at video frequencies is important, (11) may be used as a constraint in determining the constants β and I_n for the best fit of $\overline{e_{eq}^2}(\omega)$ to $\overline{e_n^2}(\omega)$.

As an example, the rms noise voltage for the Si Read diode of Fig. 2 was calculated for a bandwidth of 1 Hz and its frequency dependence is shown by the solid curve in Fig. 3. The broken curve gives the rms terminal noise voltage for the model, in 1-Hz bandwidth, with values for β and I_n chosen for a good fit to the solid curve. A third curve also shows the terminal noise voltage when β and I_n are selected with the constraint (11), i.e., the equivalent circuit predicts correct noise voltage at video frequencies.

When the Read-model approximation is used for the IMPATT diode, the constants β and I_n may be calculated from diode parameters as follows. If β is taken to be

$$\beta = \frac{L}{CR_1^2} \tag{12}$$

and (7) and (9) are used, (10) becomes

$$\overline{e_{eq}^2}(\omega) = I_n^2 \left[\frac{R_1\left(1 + \dfrac{L}{CR_1^2}\right)}{1 - \dfrac{\omega^2}{\omega_a^2}} \right]^2 \tag{13}$$

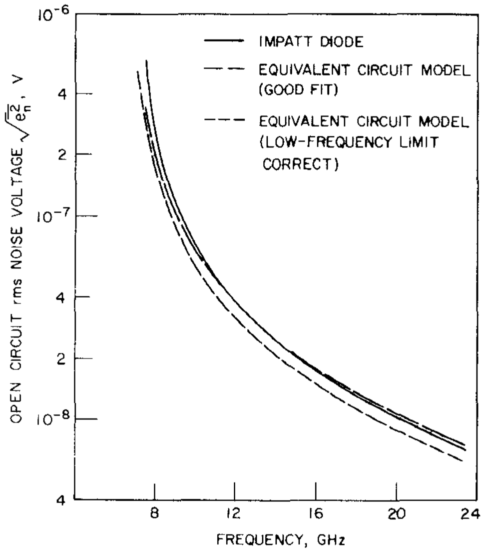

Fig. 3. Open-circuit rms noise voltage at the terminals of a Si Read diode and of the noise equivalent circuit (bandwidth = 1 Hz). For good fit, $\beta = 4$ and $I_n = 1.95 \times 10^{-10}$ A/$\sqrt{\text{Hz}}$. For correct low-frequency limit, $\beta = 4$ and $I_n = 2.32 \times 10^{-10}$ A/$\sqrt{\text{Hz}}$.

which is of the form given by (2). A comparison of (2) and (13) determines the rms noise current per unit bandwidth I_n as

$$I_n = \frac{aV_B}{\sqrt{I_{dc}}R_1\left(1 + \dfrac{L}{CR_1^2}\right)}. \tag{14}$$

An approximate noise equivalent circuit is thus determined by (12) and (14) along with (5)–(9).

The noise equivalent circuit determined with Read-model approximation is also valid at video frequencies. The low-frequency limit of noise voltage given by (2) agrees with (11) when β and I_n are given by (12) and (14), respectively. The low-frequency limit of the diode impedance given by (3) is also $R_1 + R_3$, which is equal to the space charge resistance

$$R_{sc} = \frac{l_d^2}{2\epsilon v_s A} \tag{15}$$

but neglects thermal and series resistances [16].

Application of Noise Equivalent Circuit

As an example of the use of the proposed noise equivalent circuit, the frequency dependence of the noise figure of a small-signal IMPATT-diode amplifier will be calculated. A circulator-coupled amplifier is assumed to employ the Si IMPATT diode considered earlier whose equivalent circuit elements are given in Figs. 2 and 3. Hines [13] has given the expressions for amplifier gain and noise figure using a simple model in which the diode package, transmission line, and matching arrangement taken together serve to tune out the diode reactance X_D and transform the real part of the load impedance to a value R_L close to the diode resistance R_D in magnitude. The power gain is then given by

$$G = \left(\frac{R_D - R_L}{R_D + R_L}\right)^2 \tag{16}$$

and the noise figure by

$$F = 1 + \frac{\overline{e_n^2} R_L}{(R_D + R_L)^2 G k T_0} \tag{17}$$

where $\overline{e_n^2}$ is the mean-square noise voltage per unit bandwidth, k is Boltzmann's constant, and T_0 is the absolute temperature. The amplifier noise figure[5] is calculated from the noise equivalent circuit for a fixed small-signal gain of 20 dB and is plotted in Fig. 4 as a function of frequency. The results show a broad minimum of noise

[5] The noise measure is almost identical with the noise figure as the gain and the noise figure are large.

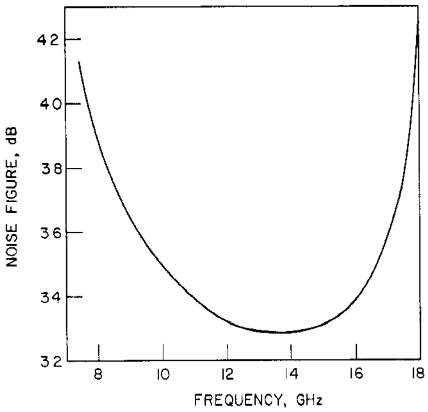

Fig. 4. Noise figure of a Si IMPATT-diode amplifier with a small-signal gain of 20 dB calculated from the noise equivalent circuit.

figure at frequencies above the frequency of maximum negative conductance.

Conclusions

It has been shown that a five lumped-element frequency-independent equivalent circuit can be constructed for IMPATT diodes. It has a driving-point impedance approximately equal to the small-signal impedance of the diode over the frequency range of interest in IMPATT-diode applications. The small-signal noise voltage across the diode can be calculated by incorporating two correlated noise sources in the equivalent circuit which are also frequency-independent. The equivalent circuit elements can be calculated by one of the following methods: 1) numerical determination to fit experimentally measured small-signal impedance (and noise voltage), 2) numerical determination to fit the small-signal impedance (and noise voltage) calculated from an accurate small-signal analysis, and 3) calculation from simple algebraic expressions given here in terms of Read-diode approximation. Frequency dependence of the noise figure of a small-signal IMPATT amplifier has been calculated to illustrate the application of the proposed equivalent circuit.

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Finite-Boundary Corrections to the Coplanar Waveguide Analysis

M. E. DAVIS, E. W. WILLIAMS, AND A. C. CELESTINI

***Abstract*—Conformal mapping calculations of impedance and effective dielectric constant are presented for coplanar waveguide (CPW) lines with finite-substrate thickness. These calculations and experimental data show a departure from the infinite dielectric approximation as the substrate thickness approaches the guide slot width. The quasi-TEM approximation is retained and calculations of static energy density within the substrate are given. This approximation agrees well with field calculations using a finite-element solution to Laplace's equation.**

Introduction

Calculations of wave impedance and effective dielectric constant of the coplanar waveguide (CPW) structure have assumed that the substrate was infinite in extent [1]. This allowed a simple conformal mapping transformation of the field patterns, using complete elliptic integrals, into a homogeneous rectangular configuration. A few investigations bore out the assumption that a finite substrate would not affect the wave propagation for simple waveguide structures [2], [3], but no extended circuitry has been reported that gives quantitative support of these assumptions. In fact, recent investigations into the CPW show marked deviation from the idealized model under some common experimental conditions [4].

As a result, a detailed calculation of the wave impedance has been carried out, still within the zeroth-order approximation of a quasi-TEM structure. By using the familiar Schwarz–Christoffel transformation of the waveguide shown in Fig. 1, the lower half of the Z plane is mapped into the rectangle in the W plane. The transformation characterizing this mapping is [1]

$$\frac{dw}{dz} = \frac{A}{(z^2 - a_1^2)^{1/2}(z^2 - b_1^2)^{1/2}} \tag{1}$$

where A is a constant to be evaluated. The boundaries of the $y=0$ line can be determined in the W plane upon integration

$$W = a + jb = \int_0^{b_1} \frac{A\,dz}{(z^2 - a_1^2)^{1/2}(z^2 - b_1^2)^{1/2}}. \tag{2}$$

The above equation is one form of an elliptic integral. The ratio a/b can be conveniently expressed in terms of tabulated complete elliptic integrals.

$$\frac{a}{b} = \frac{K(k)}{K'(k)} \tag{3}$$

where $k = a_1/b_1$, $K'(k) = K(k')$, and $k' = (1-k^2)^{1/2}$.

This is the point at which most analyses stop. By assuming a semi-infinite dielectric, in parallel with a half-space of air, the equivalent static capacitance per unit length for a pure TEM mode propagating in the line is

$$C = (\epsilon_r + 1)\epsilon_0 \frac{2a}{b} = 2(\epsilon_r + 1)\epsilon_0 \frac{K(k)}{K'(k)}. \tag{4}$$

This analysis can be extended to the case of finite-substrate thickness by mapping the substrate bottom into the W plane. This will appear approximately as an ellipse in the rectangle, as shown in Fig. 1. The mapping of complex arguments in elliptic functions is reasonably straightforward [5]. Equation (2) can be rewritten in the descriptive form

$$W = F(\phi, k) = \int_0^{\phi} \frac{d\theta}{(1 + k^2 \sin^2\theta)} \tag{5}$$

where $F(\phi, k)$ is an incomplete elliptic integral of the first kind with amplitude ϕ and modulus k. In our case $z = \sin\phi$. Since we can choose the constant A arbitrarily without affecting generality, it is given a

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